Calculation of Average Radiation Fluxes on Satellites

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It is shown that the average flux of charged particles on a satellite in a circular orbit in the radiation belt can be calculated by a method that is highly efficient in that no part of the calculation is repeated unnecessarily. The method is used to construct tables of the average daily omnidirectional fluxes of electrons and protons for orbital altitudes from 200 to 8800 km and orbital inclinations from 0° to 90°, based on the 1963 flux grid of McIlwain. It is shown that the average fluxes for elliptical orbits can easily be synthesized by hand calculation using these tables, and that the average is accurate for times longer than an apsidal precession period. The results are compared with calculations by other methods and with experiment.

I. Introduction

THE designer of a satellite needs to know the time-integrated fluxes of charged particles which the vehicle will encounter in the radiation belt. He generally obtains them from a flux map or grid by devising a computer program that simply adds up the flux contributions at suitable increments along the trajectory. This procedure would be quite satisfactory except that, for missions of long duration, it is very costly, largely because the integration method inefficiently repeats some of the most time-consuming operations many times over. The repetitions occur when the satellite visits essentially the same region of space more than once, and for long missions this is the rule.

It is shown below that in the usual case where the history of the accumulation of flux is of no interest the calculation can be organized so as to eliminate this repetition entirely. In Sec. II this calculational method is described and its physical basis is discussed. In Sec. III the conditions for the application of the method are derived. Section IV contains tables of daily average fluxes, calculated by this method for circular orbits having altitudes below 8800 km, together with a prescription for using the tables to obtain average fluxes for elliptical orbits. In Sec. V a few comparisons with results of experiment and of other calculational methods are presented. Some mathematical details are relegated to two appendices.

II. Reorganization of the Calculation

Let us suppose that the space that surrounds the earth and contains the radiation belts is divided into cells of width Δl in longitude, $\Delta\lambda$ in latitude, and ΔZ in altitude. The dimensions of the cells are to be chosen small enough so that, for practical purposes, the charged particle fluxes can be considered constant within a cell. A satellite moving on an orbit in this region would pass from cell to cell, and in the course of a long mission would visit the same cells many times. The very simple idea behind the method described here is to organize the calculation of integrated fluxes so that the costly operations of transformation to geomagnetic (B,L) coordinates[1] and flux look-up are done only once for each cell. Time-averaged fluxes (equivalent to time-integrated fluxes) are found by summing the product of the flux in each cell with the fraction of time spent there:

$$\langle F\rangle = \Sigma_{abc}F_{abc}W_{abc} \qquad (1)$$

Here F_{abc} is the flux of interest in cell (a,b,c) and W_{abc} is a weighting function equal to the fraction of time the satellite spends there. The long-time average flux $\langle F\rangle$ depends only on the apogee and perigee altitudes A and P and the orbital inclination i.

Received May 25, 1964.
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It is clear that this scheme will be advantageous only if the calculation of the weight functions is not in itself a difficult task. In what follows it will be shown that, under conditions satisfied by most orbits, simple effective weight functions can be found in the form

$$W_{abc}(i, A, P) = W_aW_b(i)W_c(A, P) \qquad (2)$$

Here W_a is the fraction of time the satellite spends in the ath longitude interval, W_b is the fraction of time it spends in the bth latitude interval, and W_c is the fraction of time it spends in the cth altitude interval.

It is somewhat surprising that weight functions of the form indicated in Eq. (2) can be used, for this implies that the samplings by the satellite of longitude, latitude, and altitude are all uncorrelated. On the other hand, it is quite obvious from the laws of mechanics that they are not. The explanation lies in the fact that the weight functions of Eq. (2) are not exact; they are approximations that give the right answer when used for long-time averaging. The reasons for their existence are to be found in the physical properties of the radiation belts. First, the variability of the fluxes with latitude and longitude is limited. This can be seen from the fact that they are adequately correlated using magnetic coordinates derived from the spherical-harmonic fit to the geomagnetic field by Jensen and Cain,[2] which contains no terms with smaller period than $\pi/3$ rad. Second, the fluxes are not known anywhere to an accuracy better than about 10%, and perhaps never will be. Consequently, the cell dimensions Δl and $\Delta\lambda$, if chosen to be the distances over which the fluxes change by an amount of the order of their uncertainty, turn out to be fairly large. Third, the fluxes are found to vary so much with time beyond 2.5 earth radii or so, depending on geomagnetic activity, that their accurate prediction by any method is problematical. This means that consideration can be restricted to the lower orbits where there is a great disparity between the periods of the satellite, the earth's rotation, and the apsidal precession. It is under these conditions that the approximate weight functions of Eq. (2) can be shown to be applicable, in all but a few excluded cases, for the calculation of average fluxes.

It is possible to give an intuitive picture of the way this comes about. Consider first a satellite in a moderately low, circular orbit. It completes many periods in a day, and so in a few days has crossed each accessible latitude many times. The distribution of these crossings among a relatively small number of longitude cells tends to become uniform, except in special cases where the orbit closes and repeats quite early, with the number of visits to each cell large enough to make the deviations between cells unimportant. Next, consider a satellite in an elliptical orbit. The apsidal precession is so slow that it can be ignored over intervals of a few days, so that each time the satellite arrives at a given latitude from a given direction it is at the same altitude. The same arguments used for circular orbits can be applied to show that the longitude sampling is effectively uniform, and therefore independent of latitude and altitude. During an apsidal precession period the orbit rotates in its plane as a rigid body, and so it is clear that each altitude on the orbit performs the same latitude sampling, which sampling is therefore independent of altitude. Thus, factored effective weight functions give average fluxes that are exactly right at multiples of the apsidal precession period and are useful approximations after one or two such periods.

III. Conditions for the Applicability of the Method

To cast the foregoing physical arguments in mathematical form and derive sufficient conditions for the applicability of the method cannot, unfortunately, be done quickly. It is to be hoped that the present treatment represents a reasonable compromise between rigor and tedium.

Consider first the assumption that any flux of interest F can be considered constant on a scale Δl or smaller in longitude. This will be taken to imply everywhere

$$\Delta l|\partial F/\partial l| \ll F \tag{3}$$

The flux function F can be expanded in an ordinary Fourier series in the longitude,

$$F = \langle F\rangle_l + \Sigma_m(C_m \cos ml + S_m \sin ml) \tag{4}$$

with

$$\left.\begin{aligned} \langle F\rangle_l &= (2\pi)^{-1}\int_0^{2\pi} F\,dl \simeq \frac{\Delta l}{2\pi}\Sigma_a F_{abc} \\ C_m &= \pi^{-1}\int_0^{2\pi} F \cos ml\,dl \simeq (\Delta l)\pi^{-1}\Sigma_a F_{abc}\cos(ma\Delta l) \\ S_m &= \pi^{-1}\int_0^{2\pi} F \sin ml\,dl \simeq (\Delta l)\pi^{-1}\Sigma_a F_{abc}\sin(ma\Delta l) \end{aligned}\right\} \tag{5}$$

The sums in Eq. (5) range over $a = 1, 2, \ldots, M = 2\pi/\Delta l$. The assumption is easily shown to imply that

$$|C_m| \cong |S_m| \cong 0 \qquad m \gtrsim M \tag{6}$$

By integrating by parts in Eq. (5) and using Eq. (3) and the fact that fluxes are non-negative, one can show that

$$\left.\begin{aligned} C_m &= (m\pi)^{-1}\int_0^{2\pi} \frac{\partial F}{\partial l}\sin ml\,dl \\ |C_m| &\le (m\pi)^{-1}\int_0^{2\pi}\left|\frac{\partial F}{\partial l}\right| dl \\ &\ll (m\pi\Delta l)^{-1}\int_0^{2\pi}|F|\,dl = \frac{2}{m\Delta l}\langle F\rangle_l \end{aligned}\right\} \tag{7}$$

with a similar bound for $|S_m|$. Equation (4) can also be put in the alternative form

$$\begin{aligned} F &= \langle F\rangle_l + \Sigma_m G_m \cos(ml - v_m) \\ G_m &= (C_m^2 + S_m^2)^{1/2} \\ v_m &= \cos^{-1}(C_m/G_m) = \sin^{-1}(S_m/G_m) \end{aligned} \tag{8}$$

and from Eqs. (6) and (7) it follows that

$$|G_m| \ll (8^{1/2}/m\Delta l)\langle F\rangle_l \qquad G_m \cong 0 \qquad m \gtrsim M \tag{9}$$

Next consider the cell weight function W_{abc} of Eq. (1). It can always be written, without loss of generality, in the form

$$W_{abc} = \Sigma_{bc} W(a,bc) W_{bc} \tag{10}$$

where $W(a,bc)$ means the fraction of time the satellite spends in the ath longitude interval, given that it is in the bth latitude and the cth altitude intervals. As noted previously, for times that are short compared with the apsidal precession period, all visits of a satellite to a given latitude from a given direction occur at essentially the same altitude. Also, the distribution in longitude of these visits is related to the distribution for similar visits to any other accessible latitude by a simple displacement in longitude, so that it suffices to consider only northward equatorial crossings. The distribution for these crossings is derived in Appendix A, where it is shown that the fraction of N northward equatorial crossings that occur in the ath longitude interval of width Δl is

$$P_a(N) = \frac{\Delta l}{2\pi} + \frac{2}{\pi N}\sum_{m=1}^{\infty}\frac{\sin(m\Delta l/2)\sin(Nmr\pi)\cos(m\psi_a)}{m\sin(mr\pi)} \tag{11}$$

$$a = 1, 2, \ldots, M$$

with

ψ_a = $[(a - \frac{1}{2})\Delta l - L_1 + (N - 1)r\pi]$
r = T_s/T_{ep}
T_s = the period of the satellite
T_{ep} = $T_e T_{pn}/(T_{pn} - T_e)$
T_e = the earth's rotation period
T_{pn} = the nodal precession period
L_1 = the longitude of the initial crossing

To generalize Eq. (11) to apply to any latitude requires only that the phase L_1 be considered to be a function of latitude and altitude (or cell numbers b and c); let this now be assumed, so that one can make the identification

$$W(a,bc) = P_a(N) \tag{12}$$

To find the conditions under which the longitude weight function is effectively uniform, one substitutes the weight functions given by Eqs. (10–12) into Eq. (1). The result is

$$\begin{aligned} \Sigma_{abc}F_{abc}W_{abc} = \Sigma_{bc}W_{bc}\{&\Sigma_a(\Delta l/2\pi)F_{abc} + \\ &2(\pi N)^{-1}\Sigma_m \sin(m\Delta l/2)\sin(Nmr\pi)\times \\ &\csc(mr\pi)m^{-1}\Sigma_a F_{abc}\cos(m\psi_a)\} \end{aligned} \tag{13}$$

where an order of summation has been interchanged. The first term in the braces of Eq. (13) is $\langle F\rangle_l$ of Eq. (5), the longitudinal average of the flux computed with a uniform weight $W_a = (\Delta_l)/2\pi$. It will next be shown that, under suitable conditions, the remaining terms in this brace become negligible compared with the first.

Inspection of the final sum in Eq. (13) shows that it can be expressed as

$$\Sigma_a F_{abc}\cos(m\psi_a) = \pi G_m(\Delta l)^{-1}\cos(mu + v_m)$$

where

$$u = (N - 1)r\pi - \tfrac{1}{2}(\Delta l) - L_1$$

using Eqs. (8) and (11). Then, by virtue of Eq. (9), it is correct to terminate the sum over m in Eq. (13) at $m = M$, and the statement to be proved can be written

$$\left|\frac{2}{(\Delta l)N\langle F\rangle_l}\sum_{m=1}^{M}\frac{G_m\sin(m\Delta l/2)\cos(mu + v_m)}{m}\,\frac{\sin(Mmr\pi)}{\sin(mr\pi)}\right| \ll 1$$

for N sufficiently large. The result would be obvious, in view of the N^{-1} dependence, were it not for the fact that the ratio $\sin(Nmr\pi)/\sin(mr\pi)$ can become as large as N in some cases, when r is a rational number. The conditions that are sought must test for, and exclude, these cases.

The period ratio r can always be written in the form

$$r = (q + \delta)^{-1} \tag{14}$$

where q is an integer and $-\frac{1}{2} < \delta \leqslant +\frac{1}{2}$. For the orbits considered here, $5 \lesssim q \lesssim 17$. By multiplying both the numerator and the denominator of Eq. (14) by successive integers $n = 1, 2, \ldots,$ (largest integer in M/q), one can generate the sequence

$$r = 1/(q + \delta) = 2/(q_2 + \delta_2) = \ldots = n/(q_n + \delta_n) = \ldots \quad (15)$$

where the q_n are integers and the δ_n satisfy $-\frac{1}{2} < \delta_n \leqslant +\frac{1}{2}$. The q_n represent the values of m for which the troublesome "large terms" occur. If any of the δ_n vanish, the factor $\sin(Nmr\pi)/\sin(mr\pi)$ equals N for those terms; otherwise it is smaller. Let

$$\epsilon = \min(|\delta_n|) \qquad N_1 = (r\epsilon)^{-1} \quad (16)$$

Then

$$|\sin(Nmr\pi)\csc(mr\pi)| \leqslant \left|\csc\left(\frac{n\pi q_n}{q_n + \delta_n}\right)\right| \leqslant (2r|\delta_n|)^{-1} \leqslant \frac{N_1}{2} \quad (17)$$

where Eqs. (14) and (15) have been used along with

$$|\sin x| \leqslant 2x/\pi \qquad 0 \leqslant x \leqslant \pi/2$$

The sum of all the "large terms" consequently has the bound

$$\frac{2}{(\Delta l)N\langle F\rangle_l}\frac{N_1}{2}\Sigma_n|m^{-1}G_m|_{m=q_n} \ll \frac{N_1\,8^{1/2}}{N(\Delta l)^2}\Sigma_n\,(q_n)^{-2} \lesssim \left(\frac{\pi^2 8^{1/2}}{6}\right)\left(\frac{N_1}{N}\right)\left(\frac{r}{\Delta l}\right)^2 \quad (18)$$

where Eq. (9) has been used.

It can be shown by entirely similar methods that the "large terms" are the only important ones in the sum, provided that $N_1 r \gtrsim 1$, which is always the case. It can therefore be asserted that the first term on the right-hand side of Eq. (13) dominates after a number of orbits N such that

$$N \gtrsim (\pi^2 8^{1/2}/6)(r/\Delta l)^2 N_1 = 68r^2 N_1 \sim N_1 \quad (19)$$

if the reasonable value $\Delta l = \pi/12$ is used.

The stated assumption that the apsidal precession can be ignored for times long enough to establish this uniform longitude sampling must be examined next. Consider a typical annular region bounded by altitudes z and $z + \Delta z$ and latitudes λ and $\lambda + \Delta\lambda$ within which the uniform longitude sampling is to occur. The visits of the satellite to this region will occur in general on some number of consecutive orbits during that portion of each apsidal precession period that is required for the orbit to drift across the region. The situation is illustrated in Fig. 1. It is necessary that the number of visits exceed N. On some occasions the satellite enters the region through the upper or lower boundary; the number of such visits will be estimated by

$$\Delta\lambda[(d\lambda/d\theta_p)(d\theta_p/dt)T_s]^{-1} = \Delta\lambda T_{pa}[2\pi T_s \sin i]^{-1}$$

where

θ_p = argument of perigee
T_{pa} = aspidal precession period

The number of other visits, where entry is through the northern or southern boundary, is estimated by

$$\Delta z[(dz/d\theta_p)_\theta(d\theta_p/dt)T_s]^{-1}$$

where θ is the argument of the satellite. An elementary calculation shows that

$$(dz/d\theta_p)_\theta \leq 0.587(A - P) \text{ for } A \leq 8800 \text{ km}$$

so the number of visits of this kind is

$$\Delta z[0.587(A - P)2\pi T_s]^{-1}\,T_{pa}$$

Requiring that the total number of visits exceed N then yields

$$(2\pi T_s)^{-1}\,T_{pa}[\Delta z/0.587\,(A - P) + \Delta\lambda/\sin i] > N \quad (20)$$

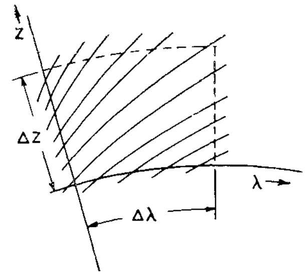

Fig. 1 Displacement of successive satellite passes through an altitude-latitude annulus due to apsidal precession. Projection on a meridian plane.

If an N can be found which will satisfy both Eqs. (19) and (20), it is sufficient to guarantee the applicability of averaging with factored weight functions. The longitude weight factor has been shown to be

$$W_a = (\Delta l)/2\pi \quad (21)$$

The latitude weight factor, being the fraction of time spent in $\lambda_2 > \lambda \geq \lambda_1$, is given by

$$W_b(i) = \pi^{-1}[\sin^{-1}(\sin\lambda_2/\sin i) - \sin^{-1}(\sin\lambda_1/\sin i)] \quad (22)$$

as a trivial application of spherical trigonometry shows for circular orbits. That this result is also correct for elliptical orbits at multiples of the apsidal precession period, and is approached as a limit after many such periods, is shown by explicit calculation in Appendix B. The absence of any z dependence in this expression shows that latitude and altitude sampling are effectively uncorrelated. Finally, the altitude weight factor for elliptical orbits, being the fraction of time spent in $z_2 \geq z \geq z_1$, is

$$W_c(A,P) = 2[\pi(A + P + 2R)]^{-1} \times \{[(A - z_1)(z_1 - P)]^{1/2} - [(A - z_2)(z_2 - P)]^{1/2}\} + \pi^{-1}\{\sin^{-1}[(A + P - 2z_1)/(A - P)] - \sin^{-1}[(A + P - 2z_2)/(A - P)]\} \quad (23)$$

where R is the radius of the earth. The derivation of Eq. (23) from elementary orbit theory is straightforward and need not be given here.

IV. Tables of Average Flux

The method described previously has been used to compute tables of average omnidirectional flux for circular orbits in the applicable altitude range. The calculations were performed using the mode 1 option of the General Dynamics/Astronautics External Radiation Flux (ERF) computer program. Flux data were obtained from the 1963 flux grid of McIlwain,[3] which is based on measurements by the Explorer XV satellite. McIlwain's flux grid was slightly modified by replacing his upper limit estimates with conservative extrapolations in some regions where no data were taken; the part of the tables seriously affected by this procedure is that part below about 400 km, where the results may be greatly in error. Time decay of the radiation belts since January 1, 1963 will cause the tabular values of average electron fluxes to be too high, particularly at the lower altitudes, but recent measurements indicate that the tables will be useful for some time yet. An example will be given in Sec. V.

The calculations were performed using an increment of longitude Δl equal to 15°. The latitude increment was chosen to be 10°, primarily because it is convenient to have it equal the increment in orbital inclination. The altitude range of the tables, 200–8800 km, is somewhat arbitrarily chosen as the region where neither high atmospheric drag nor extreme temporal variability of the radiation belts makes flux prediction questionable by this method.

The results for the average omnidirectional flux of electrons with energy greater than 0.5 Mev are given in Table 1, those for electrons with energy greater than 5.0 Mev in Table 2,

Table 1 Average for circular orbits of the omnidirectional flux of electrons with energy greater than 0.5 Mev, electrons/cm^2-day[a]

	Orbital inclination, deg				
Altitude, km	0	30	50	70	90
200	5.2 + 06	6.0 + 09	6.9 + 09	4.7 + 09	4.1 + 09
300	2.9 + 07	1.3 + 10	1.2 + 10	8.1 + 09	7.2 + 09
400	1.6 + 08	3.0 + 10	2.2 + 10	1.5 + 10	1.4 + 10
500	8.1 + 08	7.2 + 10	4.5 + 10	3.2 + 10	3.0 + 10
600	5.1 + 09	1.6 + 11	9.5 + 10	7.0 + 10	6.4 + 10
700	2.8 + 10	3.4 + 11	1.9 + 11	1.4 + 11	1.3 + 11
800	1.5 + 11	6.3 + 11	3.5 + 11	2.7 + 11	2.5 + 11
900	6.3 + 11	1.1 + 12	6.1 + 11	4.7 + 11	4.4 + 11
1000	1.7 + 12	1.7 + 12	9.7 + 11	7.6 + 11	7.1 + 11
1200	6.0 + 12	3.5 + 12	2.0 + 12	1.6 + 12	1.5 + 12
1400	1.3 + 13	6.0 + 12	3.6 + 12	2.9 + 12	2.7 + 12
1600	2.1 + 13	8.9 + 12	5.3 + 12	4.2 + 12	4.0 + 12
1800	2.8 + 13	1.1 + 13	6.8 + 12	5.5 + 12	5.1 + 12
2000	3.1 + 13	1.3 + 13	7.6 + 12	6.1 + 12	5.7 + 12
2400	3.0 + 13	1.2 + 13	7.3 + 12	5.8 + 12	5.5 + 12
2800	2.6 + 13	1.0 + 13	6.1 + 12	4.9 + 12	4.7 + 12
3200	2.0 + 13	7.9 + 12	4.8 + 12	3.8 + 12	3.6 + 12
3600	1.4 + 13	5.6 + 12	3.4 + 12	2.7 + 12	2.6 + 12
4000	9.5 + 12	3.7 + 12	2.3 + 12	1.8 + 12	1.7 + 12
4400	6.1 + 12	2.4 + 12	1.5 + 12	1.2 + 12	1.1 + 12
4800	4.1 + 12	1.6 + 12	9.8 + 11	7.9 + 11	7.4 + 11
5200	2.7 + 12	1.0 + 12	6.4 + 11	5.1 + 11	4.8 + 11
5600	1.6 + 12	6.2 + 11	3.9 + 11	3.1 + 11	2.9 + 11
6000	9.5 + 11	3.7 + 11	2.4 + 11	1.9 + 11	1.8 + 11
6400	5.8 + 11	2.4 + 11	1.6 + 11	1.2 + 11	1.2 + 11
6800	3.5 + 11	1.6 + 11	1.2 + 11	8.8 + 10	8.1 + 10
7200	2.2 + 11	1.2 + 11	9.6 + 10	6.9 + 10	6.3 + 10
7600	1.5 + 11	9.3 + 10	8.1 + 10	5.6 + 10	5.1 + 10
8000	1.0 + 11	7.7 + 10	7.2 + 10	4.9 + 10	4.4 + 10
8800	6.9 + 10	5.8 + 10	5.2 + 10	3.6 + 10	3.2 + 10

[a] $X + 0Y$ means $X \cdot 10^Y$.

Table 2 Average for circular orbits of the omnidirectional flux of electrons with energy greater than 5.0 Mev, electrons/cm^2-day[a]

	Orbital inclination, deg				
Altitude, km	0	30	50	70	90
200	1.0 + 03	6.7 + 07	9.0 + 07	6.0 + 07	5.3 + 07
300	1.2 + 04	1.7 + 08	1.6 + 08	1.0 + 08	9.5 + 07
400	1.7 + 05	4.0 + 08	2.9 + 08	2.0 + 08	1.8 + 08
500	2.4 + 06	1.0 + 09	6.1 + 08	4.4 + 08	4.0 + 08
600	4.4 + 07	2.7 + 09	1.5 + 09	1.1 + 09	1.0 + 09
700	3.2 + 08	6.7 + 09	3.7 + 09	2.8 + 09	2.6 + 09
800	2.4 + 09	1.5 + 10	8.4 + 09	6.5 + 09	6.0 + 09
900	1.5 + 10	2.8 + 10	1.6 + 10	1.2 + 10	1.2 + 10
1000	5.2 + 10	4.8 + 10	2.8 + 10	2.2 + 10	2.0 + 10
1200	2.0 + 11	9.6 + 10	5.7 + 10	4.6 + 10	4.3 + 10
1400	4.1 + 11	1.6 + 11	9.7 + 10	7.8 + 10	7.3 + 10
1600	6.4 + 11	2.2 + 11	1.4 + 11	1.1 + 11	1.0 + 11
1800	7.3 + 11	2.6 + 11	1.6 + 11	1.3 + 11	1.2 + 11
2000	7.1 + 11	2.6 + 11	1.6 + 11	1.3 + 11	1.2 + 11
2400	4.7 + 11	1.8 + 11	1.1 + 11	8.7 + 10	8.2 + 10
2800	2.7 + 11	1.0 + 11	6.4 + 10	5.1 + 10	4.8 + 10
3200	1.5 + 11	5.8 + 10	3.6 + 10	2.9 + 10	2.7 + 10
3600	7.6 + 10	3.2 + 10	1.9 + 10	1.6 + 10	1.4 + 10
4000	3.7 + 10	1.8 + 10	1.1 + 10	8.6 + 09	8.0 + 09
4400	2.4 + 10	1.3 + 10	7.7 + 09	6.1 + 09	5.7 + 09
4800	2.6 + 10	1.2 + 10	7.1 + 09	5.6 + 09	5.3 + 09
5200	2.5 + 10	1.0 + 10	6.4 + 09	5.1 + 09	4.8 + 09
5600	1.8 + 10	7.7 + 09	4.7 + 09	3.8 + 09	3.5 + 09
6000	9.9 + 09	5.0 + 09	3.1 + 09	2.4 + 09	2.3 + 09
6400	6.5 + 09	3.7 + 09	2.3 + 09	1.8 + 09	1.6 + 09
6800	5.5 + 09	3.2 + 09	2.0 + 09	1.5 + 09	1.4 + 09
7200	5.1 + 09	2.9 + 09	1.8 + 09	1.4 + 09	1.3 + 09
7600	4.7 + 09	2.7 + 09	1.7 + 09	1.3 + 09	1.2 + 09
8000	4.4 + 09	2.4 + 09	1.6 + 09	1.2 + 09	1.1 + 09
8800	3.5 + 09	1.9 + 09	1.2 + 09	9.3 + 08	8.6 + 08

[a] $X + 0Y$ means $X \cdot 10^Y$.

and those for protons with energy between 40 and 110 Mev in Table 3. Interpolation of the tabular values in altitude should present no problems. Interpolation in energy is best performed on the logarithm of the fluxes for electrons; for protons, the energy spectrum is unfortunately rather dependent upon position.

The tables can be used to calculate the average fluxes on elliptical orbits which fall within the tabulated altitude range and which satisfy the criteria of Eqs. (19) and (20). It is only necessary to use Eq. (23) to calculate altitude weight functions for regions Δz about the tabular values, multiply them by the tabulated values for circular orbits, and add up the results. Experience indicates that this simple integration technique is adequate, and that no special care is required at the end points or at places where the table spacing changes. It is even possible to obtain good results using, instead of the rather complicated Eq. (23), the differential form of the altitude weight factor

$$W_c(A, P) = 2(z + R)\Delta z[\pi(A + P + 2R)]^{-1}[(A - z)(z - P)]^{-1/2} \quad (24)$$

but here some care must be taken near the end points.

As an illustration of the efficiency of this method, it may be noted that the total cost of computing the tables was roughly equal to that of computing the time-integrated flux on one elliptical orbit by a direct method.

V. Comparison with Other Calculational Methods and with Experiment

It has not been possible to give a general account of the accuracy of the averages calculated by this method, nor to estimate the magnitude of the deviations from the average. Some idea may be gained from a few illustrative examples where the accumulation of flux has also been calculated by

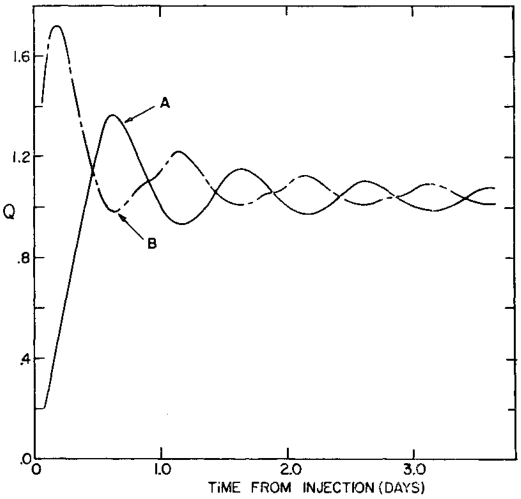

Fig. 2. Comparison of long-term and current average proton fluxes for a circular orbit at 1000 km and 30° inclination. Initial positions: *A*, 30°N, 305°E; *B*, 30°S, 326°E.

the ERF code, operating with the same flux grid in a different mode, in proper time sequence.

Let the ratio of the "current average" to the average calculated from the tables be defined by

$$Q(t) = \langle F \rangle^{-1} t^{-1} \int F(t')dt'$$

Figure 2 shows the quantity Q for a circular orbit at 1000 km altitude and 30° inclination, for the proton flux, and for two starting points corresponding roughly to Cape Kennedy and the South Atlantic. The criteria of Sec. III, applied to this orbit, predict that the tabular average should be accurate

Table 3 Average for circular orbits of the omnidirectional flux of protons with energy between 40 and 110 Mev, protons/cm²-day[a]

	Orbital inclination, deg				
Altitude, km	0	30	50	70	90
200	1.1 + 04	4.7 + 05	1.2 + 06	1.4 + 06	1.1 + 06
300	2.8 + 04	8.8 + 05	1.5 + 06	1.6 + 06	1.3 + 06
400	7.1 + 04	1.6 + 06	2.1 + 06	2.0 + 06	1.7 + 06
500	1.6 + 05	2.9 + 06	3.2 + 06	2.7 + 06	2.3 + 06
600	4.6 + 05	4.8 + 06	4.5 + 06	3.7 + 06	3.2 + 06
700	1.3 + 06	7.9 + 06	6.5 + 06	5.2 + 06	4.5 + 06
800	3.8 + 06	1.2 + 07	9.1 + 06	7.1 + 06	6.3 + 06
900	9.3 + 06	1.9 + 07	1.3 + 07	1.0 + 07	9.1 + 06
1000	2.0 + 07	2.8 + 07	1.8 + 07	1.4 + 07	1.3 + 07
1200	6.1 + 07	5.6 + 07	3.5 + 07	2.7 + 07	2.5 + 07
1400	1.3 + 08	1.0 + 08	6.1 + 07	4.8 + 07	4.4 + 07
1600	2.4 + 08	1.6 + 08	9.5 + 07	7.4 + 07	6.9 + 07
1800	3.6 + 08	2.2 + 08	1.3 + 08	1.0 + 08	9.7 + 07
2000	5.0 + 08	2.8 + 08	1.7 + 08	1.3 + 08	1.2 + 08
2400	7.3 + 08	3.8 + 08	2.3 + 08	1.8 + 08	1.7 + 08
2800	8.8 + 08	4.2 + 08	2.5 + 08	2.0 + 08	1.9 + 08
3200	9.0 + 08	4.1 + 08	2.5 + 08	2.0 + 08	1.8 + 08
3600	8.2 + 08	3.6 + 08	2.2 + 08	1.7 + 08	1.6 + 08
4000	6.9 + 08	3.0 + 08	1.8 + 08	1.4 + 08	1.3 + 08
4400	5.4 + 08	2.3 + 08	1.4 + 08	1.1 + 08	1.0 + 08
4800	3.8 + 08	1.7 + 08	1.0 + 08	8.2 + 07	7.7 + 07
5200	2.7 + 08	1.2 + 08	7.6 + 07	6.0 + 07	5.6 + 07
5600	2.1 + 08	9.8 + 07	6.0 + 07	4.7 + 07	4.4 + 07
6000	1.8 + 08	8.6 + 07	5.2 + 07	4.1 + 07	3.8 + 07
6400	1.8 + 08	8.1 + 07	5.0 + 07	3.9 + 07	3.7 + 07
6800	1.8 + 08	8.1 + 07	4.9 + 07	3.9 + 07	3.6 + 07
7200	1.9 + 08	8.0 + 07	5.0 + 07	3.9 + 07	3.7 + 07
7600	1.9 + 08	7.7 + 07	4.8 + 07	3.8 + 07	3.5 + 07
8000	1.7 + 08	6.8 + 07	4.2 + 07	3.4 + 07	3.1 + 07
8800	1.0 + 08	4.1 + 07	2.6 + 07	2.0 + 07	1.9 + 07

[a] $X + 0Y$ means $X \cdot 10^Y$.

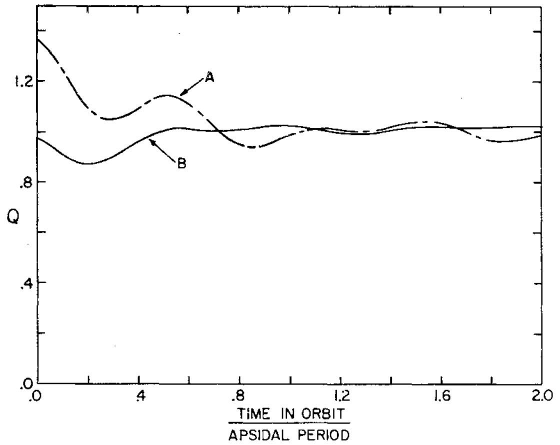

Fig. 3 Comparison of long-term and current average proton fluxes on two elliptic orbits; orbit *A*: *P* = 350 km, *A* = 1200 km, i = 98°, initial position 34°N, 61°E; orbit *B*: *P* = 648 km, *A* = 3426 km, *i* = 30°, initial position 14°N, 178°E.

after about one day; inspection of Fig. 2 shows that neither curve deviates from 1 by more than 20% after the first day. The limit of $Q(t)$ as t becomes infinite gives the ratio of the directly calculated long-term average to the tabular one. It can be found by fitting t^{-1} envelopes to the curve, and turns out to be 1.016, which is somewhat better agreement than should be expected with two-place tables. This demonstrates, for one thing, the sufficiency of the mesh size used in the construction of the tables.

Figure 3 shows $Q(t)$ for two elliptical orbits. Both satisfy the criteria of Sec. III, so that the tabular average is expected to be accurate after one apsidal precession period. In fact, the tabular average is accurate to 1.5% for orbit A and 2.2% for orbit B at this time, with deviations from the average during the first precession period less than 50% and less than 5% during the second.

The really significant test of a prediction method is comparison with experiment, of course. The single example of this kind that is currently available is shown in Fig. 4, which presents the daily rate of accumulation of flux of electrons with energy greater than about 4 Mev as measured on the Relay I satellite during the latter half of 1963 (instrumental difficulties make data from the first half of the year less reliable).[4] The tabular average, obtained by interpolation in energy from Tables 1 and 2, is also shown. The Relay I orbit satisfies the criteria of Sec. III, so that this average should be accurate after an apsidal precession period of about 280 days. In fact, a complete sampling can be obtained in half a period, if the starting point is properly chosen. This is the case here, so that the prediction and experiment should be comparable. It will be seen that the tabular average is 40% below the measured value, and in view of the uncertainties associated with both the experimental data, where the threshhold energy is not well-known, and with the interpolation, this can be considered quite adequate agreement.

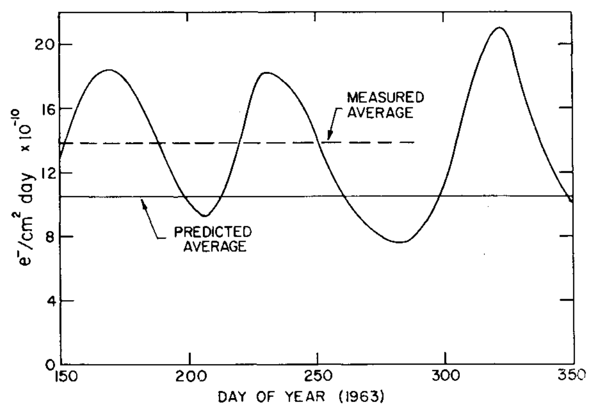

Fig. 4 Measured daily fluxes of electrons with >4.0 Mev on Relay I compared with the predicted long-term average; *A* = 6825 km, *P* = 1339 km, *i* = 47.5°.

In making this last comparison, no attempt has been made to adjust the tabular average for time decay of the artificial electron radiation belt. Recent determinations indicate that this decay is slow, with an exponential mean life of the order of a year or more. Consequently, a flux grid of epoch 1963.0 and the tables derived from it will be useful for some time to come.

Appendix A: Derivation of the Longitude Distribution of Northward Equatorial Crossings

The northward equatorial crossings of a satellite occur at a sequence of east longitudes L_n given by

$$L_n = L_1 - 2\pi n T_s/T_{ep} \qquad n = 0, 1, 2, \ldots \tag{A1}$$

where L_1 is the east longitude of the first crossing, and T_s and T_{ep} are defined following Eq. (11). The study of the distribution of the L_n among longitude cells is facilitated by the introduction of the function

$$\left.\begin{aligned} S(y, L) &= \begin{cases} 1 & y \leq L < y + \Delta l \\ 0 & \text{elsewhere in } 0 \leq L < 2\pi \end{cases} \\ S(y + 2\pi, L) &= S(y, L) \qquad \text{all } L \end{aligned}\right\} \tag{A2}$$

This function counts the times that L falls in a cell of width Δl with western edge at y, and by virtue of its periodicity effectively reduces L to $0 \leq L < 2\pi$. In terms of S it is easy to express the fraction of N total crossings that occur in this cell:

$$P(y, N) = N^{-1} \sum_{n=0}^{N-1} S(y, L_n) \tag{A3}$$

with L_n given by Eq. (A1).

Equation (A3) can be transformed to a more useful form by calculating its Laplace transform with respect to y. Let

$$p(x, N) = \int_0^\infty e^{-xy}\, P(y,N)\, dy$$

Then it is easily shown that[5]

$$\begin{aligned} p(x,N) &= [Nx(1 - e^{-2\pi x})]^{-1}(e^{x\Delta l} - 1)\Sigma_n e^{-xL_n} \\ &= e^{-xL_1}(e^{x\Delta l} - 1)(e^{2\pi r N x} - 1)[Nx(1 - e^{-2\pi x})(e^{2\pi r x} - 1)]^{-1} \end{aligned} \tag{A4}$$

where

$$r = T_s/T_{ep}$$

and Eq. (A1) has been used. The only singularities of $p(x,N)$, as a function of x, are simple poles at $x = m(-1)^{1/2}$, $m = 0, \pm 1, \pm 2, \ldots$, and its inverse is the sum of the residues of $e^{xy}p(x,N)$ at these poles.[5] Putting $y = a\Delta l$, one finds, after some obvious manipulation,

$$P_a(N) \equiv P(a\Delta l, N) = \frac{(\Delta l)}{2\pi} + \frac{2}{\pi N} \sum_{m=1}^{\infty} \frac{\sin(m\Delta l/2)\sin(Nmr\pi)\cos(m\psi_a)}{m \sin(mr\pi)} \tag{A5}$$

in a notation defined following Eq. (11).

Appendix B: Latitude Weight Factor for Elliptic Orbits

The latitude λ and the argument θ of a satellite are related by

$$\sin\lambda = \sin i \sin\theta \tag{B1}$$

The equation of an elliptical orbit is

$$\rho = a(1 - E^2)[1 + E\cos(\theta - \theta_p)]^{-1} \quad \text{(B2)}$$

where

ρ = the geocentric distance
a = the semimajor axis
E = the eccentricity

The latitude weight function is the fraction of time that the satellite spends in the latitude interval of interest·

$$w(\lambda_2, \lambda_1, \theta_p) = T_s^{-1} \int_{\theta_1}^{\theta_2} \frac{d\theta}{(d\theta/dt)}$$

$$\theta_2 = \sin^{-1}(\csc i \sin\lambda_2) \quad \text{(B3)}$$

$$\theta_1 = \sin^{-1}(\csc i \sin\lambda_1)$$

where crossings in only one direction are counted and Eq. (B1) has been used. Elementary orbit theory shows that

$$d\theta/dt = 2\pi a^2(1 - E^2)^{1/2}T_s^{-1}\rho^{-2} \quad \text{(B4)}$$

so that

$$w(\lambda_2, \lambda_1, \theta_p) = (2\pi)^{-1}(1 - E^2)^{3/2}\int_{\theta_1}^{\theta_2} [1 + E\cos(\theta - \theta_p)]^{-2}d\theta$$

upon using Eqs. (B4) and (B2) in (B3).

It will be shown that, when $w(\lambda_2, \lambda_1, \theta_p)$ is averaged over a complete apsidal precession period, the result is in agreement with Eq. (22):

$$W = (2\pi)^{-1} \int_0^{2\pi} w(\lambda_2, \lambda_1, \theta_p)\, d\theta_p$$

$$= (2\pi)^{-2}(1 - E^2)^{3/2}\int_0^{2\pi} d\theta_p \int_{\theta_1}^{\theta_2} [1 + E\cos(\theta - \theta_p)]^2 d\theta$$

The integrations are elementary, requiring only a little care in selecting the proper branch of a multiple-valued function, and yield

$$W = (\theta_2 - \theta_1)/2\pi$$

This is just half of the result given in Eq. (22), where crossings in both directions have been counted.

Averages for elliptical orbits calculated using Eq. (22) are therefore correct at one apsidal period, and this is also obviously the case at integral multiples of this period. The flux sampled by the satellite is a periodic function of time, with period T_{pa}, so that it can be written

$$F(t) = \langle F\rangle + [\text{terms in } \cos(2\pi kt/T_{pa}) \text{ and } \sin(2\pi kt/T_{pa})]$$

Then the "current average" flux is

$$\langle F\rangle_c = t^{-1}\int_0^t F(t')\,dt'$$

$$= \langle F\rangle + t^{-1}\,[\text{terms in } \sin(2\pi kt/T_{pa}) \text{ and } \cos(2\pi kt/T_{pa})]$$

Hence, the long-term average is the limit of the current average as t becomes infinite, and the convergence rate is like t^{-1}.

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